

Fiber Vector Modesolver - Improvements to the Efficient 4x4 Matrix Method

Steven R. A. Dods

*Optiwave Systems, 7 Capella Ct.
Ottawa, Ontario, K2E 7X1, Canada
steve.dods@optiwave.com*

Abstract: A numerical technique for finding all electromagnetic field components of guided modes on cylindrical structures consisting of concentric layers of loss-less dielectric is implemented by the method of Yeh and Lindgren. This paper reports three improvements to this method.

©2005 Optical Society of America

OCIS codes: (060.2310) Fiber Optics; (000.4430) Numerical Approximation and Analysis; (230.7370) Waveguides

1. Introduction

Many kinds of optical fiber can be described, or in the case of graded index fibers, approximated by, a series of concentric layers of loss-less dielectric. When the index contrast in the structure is small, it is common to use the scalar wave equation to obtain the linearly-polarized modes (LP modes). However, when the index contrast becomes larger, the LP approximation becomes inaccurate, and a full vector analysis is necessary [1]. Within any one of the concentric layers, the field components are all linear combinations of Ordinary and Modified Bessel functions, and the solution is a matter of matching the tangential field components of adjacent layers, leading to the solution of a linear system. There are some choices to make about the best way to arrange the linear system (see Ref. 2 for a complete treatment) but probably the best choice for a general, quick algorithm is the method of Yeh and Lindgren [3]. The method has intuitive appeal in its similarity to the transfer matrix method used for the analysis of planar waveguides [4]. Both methods use a matrix to express the fields at one side of a layer given their values at the other side, and the whole structure is then characterized by cascading the layers by matrix multiplication. For vector fields in fibers, there is no natural separation into TE and TM polarization, so one must include the two tangential components for both electric and magnetic field simultaneously. In total there are four transverse components at the layer boundaries, so the transfer matrix in fiber is a 4x4 matrix, instead of 2x2.

This paper describes three ways the 4x4 matrix method for vector modes in fibers can be improved to make the algorithm easier to implement, faster in execution time, and more reliable in the finding of modes, particularly in cases where the modes are almost degenerate. The first improvement involves a reformulation of the basic equations to make a real-valued numerical implementation. Since it is loss-less modes on a fiber of loss-less material that is being sought, there is no reason to include complex numbers in the formulation. Real-valued implementation uses less computer resources. Second, the complete transfer across one layer boundary requires a matrix inverse. It is better to find this inverse analytically, rather than rely on numerical inversion at each stage. In this paper it is shown that the layer matrix can be decomposed into two 4x4 matrices, and the new matrices are of a form where it is easy to find the inverse by inspection, thereby finding the analytic solution to the original inversion problem. Third, it is usual to construct the dispersion function (the zeros of which are located at the modal indices) as the determinant of the 4x4 matrix system. While this is theoretically correct, it is not the most intuitive prescription. Worse, the determinant is not the most convenient construction for locating the zeros, particularly in the case where two zeros are very close to each other. This paper shows an eigen-value analysis that splits the dispersion function into two functions. This splitting resolves the almost degenerate pairs and helps the simple-minded computer algorithms to find the zeros of the dispersion function (i.e. the modes) more reliably.

2. Real-valued formulation

The time independent Maxwell curl equations with a positive time convention ($e^{+j\omega t}$) are

$$\nabla \times \mathbf{E} = -j\omega\mu_0\mathbf{H} \qquad \nabla \times \mathbf{H} = j\omega\varepsilon_0\varepsilon(r)\mathbf{E} \qquad (1)$$

The factor j does not suit the current purpose, so make the substitution

$$\mathbf{h} = -j\eta_0\mathbf{H} \quad (2)$$

where $\eta_0 = \sqrt{\mu_0 / \varepsilon_0}$ is the free space impedance, so that \mathbf{h} and \mathbf{E} have the same units. Then

$$\nabla \times \mathbf{E} = k\mathbf{h} \quad \nabla \times \mathbf{h} = k\varepsilon(r)\mathbf{E} \quad (3)$$

Where $k = \omega\sqrt{\mu_0\varepsilon_0}$. For finding electromagnetic fields in a cylindrical geometry it is convenient to use cylindrical coordinates r , ϕ , and z , and Debye potentials parallel to the axis of rotation, $\hat{\mathbf{z}}$ [2]:

$$\mathbf{E} = \nabla \times (\hat{\mathbf{z}}\psi) + \frac{1}{k\varepsilon(r)} \nabla \times \nabla \times (\hat{\mathbf{z}}\phi) \quad (4)$$

$$\mathbf{h} = \nabla \times (\hat{\mathbf{z}}\phi) + \frac{1}{k} \nabla \times \nabla \times (\hat{\mathbf{z}}\psi) \quad (5)$$

The potentials themselves are solutions of the scalar Helmholtz equation, and the particular solution is found by observing the boundary conditions imposed by physical considerations on \mathbf{E} and \mathbf{h} . The potentials are supposed to form modes, so a solution where the variables are separated is appropriate

$$\psi(r, \phi, z) = \psi_\nu(r) \exp[j(\nu\phi - \beta z)] \quad (6)$$

and a similar expression applies for the other Debye potential, ϕ . For regions where $\varepsilon(r)$ is constant, the radial functions are linear combinations of Bessel functions of integer order ν . In any given layer,

$$\phi_\nu(r) = AJ_\nu(ur) + BY_\nu(ur) \quad \psi_\nu(r) = CJ_\nu(ur) + DY_\nu(ur) \quad (7)$$

where $u = \sqrt{k^2\varepsilon - \beta^2}$. In layers where the propagation constant squared is larger than $k^2\varepsilon$, the Bessel functions J and Y are replaced by the Modified Bessel functions I and K , respectively. Equations (6) and (7) are substituted in (4) and (5). It is the tangential components ϕ and z that are needed explicitly, since it is the tangential components of the electric and magnetic fields that should match at layer boundaries. These field components are related to the coefficients by a 4x4 matrix

$$\mathbf{f} = \begin{bmatrix} E_z \\ h_\phi \\ h_z \\ E_\phi \end{bmatrix} = \begin{bmatrix} u^2 J_\nu(ur)/k\varepsilon & u^2 Y_\nu(ur)/k\varepsilon & 0 & 0 \\ -uJ'_\nu(ur) & -uY'_\nu(ur) & \nu J_\nu(ur)/r & \nu Y_\nu(ur)/r \\ 0 & 0 & u^2 J_\nu(ur)/k & u^2 Y_\nu(ur)/k \\ \nu J_\nu(ur)/\varepsilon r & \nu Y_\nu(ur)/\varepsilon r & -uJ'(ur) & -uY'(ur) \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} \quad (8)$$

where $n = \beta / k$ is the modal index, and the common factor $\exp[j(\nu\phi - \beta z)]$ is suppressed. A similar equation to (8) applies in adjacent layers, with different constants A, B, C, D . Given the constants in one layer, the constants in the adjacent layer can be found by solving the linear system created by the field matching condition. The \mathbf{f} found by the two matrices should be the same field vector at the boundary between layers. The difference between this formulation and that in Ref 2 and 3 is that this formulation uses real numbers only.

3. Solution of the linear system

The order of the components in the vector \mathbf{f} is chosen to help factor the matrix in (8) into two matrices as $\mathbf{Ff} = \mathbf{Dc}$ where $\mathbf{c} = [A, B, C, D]^T$ and

$$\mathbf{F} = \begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & -1 & nvk/ru^2 & 0 \\ 0 & 0 & 1 & 0 \\ nvk/ru^2 & 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} u^2 J'_\nu/k & u^2 Y'_\nu/k & 0 & 0 \\ uJ'_\nu & uY'_\nu & 0 & 0 \\ 0 & 0 & u^2 J_\nu/k & u^2 Y_\nu/k \\ 0 & 0 & uJ'_\nu & uY'_\nu \end{bmatrix} \quad (9)$$

The advantage of this factoring is that it is easy to find the expressions for \mathbf{F}^{-1} and \mathbf{D}^{-1} . \mathbf{F}^{-1} is found by two row reductions and \mathbf{D}^{-1} by inverting the 2x2 matrices in the block diagonal. These matrices are required to find the coefficients in the next layer. If \mathbf{c}_m are the coefficients in the m th layer, then the coefficients in the next layer is found from the matrix product

$$\mathbf{c}_{m+1} = \mathbf{D}_{m+1}^{-1}(r_m) \mathbf{F}_{m+1}(r_m) \mathbf{F}_m^{-1}(r_m) \mathbf{D}_m(r_m) \mathbf{c}_m \quad (10)$$

The use of the factored matrix means four matrix multiplications instead of two are required to cross the boundary, but it avoids having to find the matrix inverse numerically.

4. Dispersion equation

The matrix to describe the whole multilayer can be written as one 4x4 matrix, the product of all layers, one layer of which is shown in (10). The dispersion equation is usually constructed by setting the determinant of this 4x4 matrix product to 0, but this is not a good choice from the numerical point of view. In the limit of low index contrast, the modal indices can be very close to each other, making them hard to find by numerical methods, which tend to skip over closely spaced zeros. A more convenient condition comes from using rectangular matrices at the first and last layers, which are special cases. In the first layer, which contains $r = 0$, coefficients B_0 and D_0 must be zero, so the first matrix, \mathbf{D}_0 , is 4 x 2 consisting of the first and third columns of the \mathbf{D} matrix in (9). Similarly, in the last layer, it is the coefficients A_M and C_M that must go to zero, since those contain solutions I_ν with unphysical growth. The last matrix \mathbf{D}_M^{-1} will be of size 2 x 4, and consist of the first and third rows of \mathbf{D}^{-1} . The product of all these matrices will be of size 2x2, which we call $\mathbf{S}(n)$. (The dependence on n is written explicitly since it is the modal index to be found). A mode is obtained at a modal index n when a non-trivial solution is found to

$$\mathbf{0} = \mathbf{S}(n) \begin{bmatrix} A_0 \\ C_0 \end{bmatrix} \quad (11)$$

The condition $\det(\mathbf{S}) = 0$ is sufficient for a non-trivial solution, but an alternative is to consider the vector $[A_0, C_0]^T$ as an eigenvector of \mathbf{S} with eigenvalue, λ , equal to 0. The eigenvalues of $\mathbf{S}(n)$ are

$$\lambda_1(n) = T/2 + \sqrt{T^2/4 - \det(\mathbf{S})} \quad \text{and} \quad \lambda_2(n) = T/2 - \sqrt{T^2/4 - \det(\mathbf{S})} \quad (12)$$

Where $T = S_{11} + S_{22}$. A mode exists when either $\lambda_1(n) = 0$ or $\lambda_2(n) = 0$. The advantage of searching for zeros of the two functions (12) is that the functions $\lambda_1(n)$ and $\lambda_2(n)$ have zeros that are widely spaced apart, and are more easily found than the zeros of the single function that comes from setting the determinant to zero.

References

- [1] K. Okamoto, *Fundamentals of Optical Waveguides*, (Academic Press, San Diego, 2000)
- [2] C. Tsao, *Optical Fibre Waveguide Analysis*, (Oxford University Press, 1992), Part III
- [3] C. Yeh and G. Lindgren, "Computing the propagation characteristics of radially stratified fibers: an efficient method", *Applied Optics* **16**(2) p483-493 (1977)
- [4] J. Chilwell and I. Hodgkinson, "Thin-films field-transfer matrix theory of planar multilayer waveguides and reflection from prism-loaded waveguide", *Journal of the Optical Society of America A*, **1** p742-753 (1984)